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Studying the blood plasma flow past a red blood cell, with the mathematical method of Kelvin's transformation

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***Abstract*–A mathematical tool, namely the Kelvin transformation, has been employed in order to derive analytical expressions for important hydrodynamic quantities, aiming to the understanding and the study of the blood plasma flow past a Red Blood Cell (RBC). These quantities are the fluid velocity, the drag force exerted on the cell and the drag coefficient. They are obtained by employing the stream function ψ which describes the Stokes flow past a fixed cell. The RBC, being a biconcave disk, has been modelled as an inverted prolate spheroid. The stream function is given as a series expansion in terms of Gegenbauer functions, which converge fast. Therefore the first term of the series suffices for the derivation of simple and ready to use expressions.**

Index Terms – mathematical model, Kelvin transformation, Stokes flow, red blood cell, settling velocity.

I. Introduction

Stokes flow has been employed for describing the flow of many bio-fluids, such as the blood serum [1, 2, 3]. This is mainly due to the physical characteristics of the fluid, (density, viscosity, etc.), the low fluid velocity and the small size of the presented "particles". Blood plasma flow past a red blood cell has been modelled as Stokes flow past a solid inverted prolate spheroid [4]. The stream function $\psi_\alpha(\mathbf{r}')$ has been analytically obtained by employing the Kelvin transformation and it is given as a series expansion of Gegenbauer functions. Kelvin inversion has also been employed for solving exterior problems with Dirichlet and with Neumann boundary conditions [5, 6, 7, 8].

In the present work, we exploit this concept in order to calculate the velocity field of the fluid, the drag force F_z exerted on the cell and the drag coefficient [9]. Furthermore we calculate the coefficient $\mathrm{Re}\, C_D$, with Re being the Reynolds number characterising the flow, which in our case is Re << 1. Since the stream function is given through a fast converging series expansion, the first term of the series seems to be adequate, in order to gain simply "closed" formulas for the above mentioned quantities revealing their behaviour.

The structure of the manuscript is as follows. In Section II we describe the methodology we followed for the derivation of the stream function. The velocity components, the drag force and the drag coefficient are derived in Section III, where the obtained results are explained and discussed.

II. Statement of the Problem

In this model, we consider the axisymmetric Stokes flow around an inverted prolate spheroid [4]. We assume a uniform velocity with magnitude U, which is parallel to the x_3-axis in the negative direction. We also consider the inverted prolate spheroid as being a stationary and isolated solid, that is having its centre at the origin of a Cartesian coordinate system (x_1, x_2, x_3), as it shown in Figure 1.

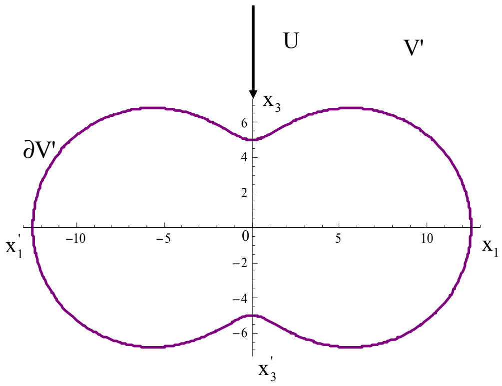

Figure 1. Stokes flow around an inverted prolate spheroid.

The problem is then defined by the equations (1)-(4) as follows

$$E'^4 \psi_\alpha(\mathbf{r}') = 0, \mathbf{r}' \in V', \quad (1)$$

$$\psi_\alpha(\mathbf{r}') = 0, \mathbf{r}' \in \partial V', \quad (2)$$

$$\frac{\partial \psi_\alpha(\mathbf{r}')}{\partial n} = 0, \mathbf{r}' \in \partial V', \quad (3)$$

$$\psi_\alpha \to \frac{1}{2} \varpi'^2 U, r' \to +\infty, \quad (4)$$

where V' is the exterior fluid domain, $\partial V'$ denotes the surface of the inverted prolate spheroid, E'^2 is the Stokes operator, $E'^4 = E'^2 \circ E'^2$ is the Stokes bistream operator, $\mathbf{r'}$ is the position vector, $\psi_\alpha(\mathbf{r'})$ is the stream function, U is the fluid velocity and ϖ' is the radial cylindrical coordinate. On the surface of the inverted prolate spheroid we impose a no slip condition (2). We also assume that the inverted prolate spheroid (RBC) is impenetrable (3), which is true in pathological situations and finally, since the fluid extends indefinitely towards all the directions, the asymptotic relation (4) has to be satisfied.
The methodology for solving (1)-(4) is as follows. First, we express the problem at hand in the inverted prolate coordinate system (τ', ζ'), [10]. Then, by employing Kelvin's inversion theorems for the Stokes operators, given by Dassios in [11], we transform the problem in the equivalent one, which is now expressed in the prolate coordinate system (τ, ζ) [10], with τ and ζ expressing the "radial" and the angular coordinates, respectively, with $\tau \geq 1$ and $-1 \leq \zeta \leq 1$.

We solve the problem in the prolate coordinate system, where the stream function semiseparates variables [12]. Using reversely the Kelvin's transformation theorems [11], we obtain the stream function $\psi_\alpha(\tau', \zeta')$ of the problem at hand, which is given in terms of the prolate coordinates as follows.

$$\psi_\alpha(\tau', \zeta') = \frac{b^3}{c^3 \sqrt{\tau^2 + \zeta^2 - 1}^{\,3}} \sum_{n=1}^{\infty} g_{2n}(\tau) G_{2n}(\zeta), \tag{5}$$

where $b > 0$ is the radius of the sphere of inversion, $c > 0$ is the semifocal distance and

$$g_2(\tau) = A_2 G_2(\tau) - \frac{9bcU}{5} H_2(\tau) - bcU G_1(\tau) + E_2 G_4(\tau) - \frac{6bcU}{5} H_4(\tau), \tag{6}$$

$$g_{2n}(\tau) = A_{2n} G_{2n}(\tau) + \frac{bcU}{2}(-w_{n-1} e_{n-1}^2 - w_n d_n^2) H_{2n}(\tau) + \frac{bcU}{2} w_{n-1} e_{n-1} d_{n-1} H_{2n-2}(\tau) + E_{2n} G_{2n+2}(\tau) + \frac{bcU}{2} w_n d_n e_n H_{2n+2}(\tau), \; n \geq 2, \tag{7}$$

where A_n, E_n, w_n, e_n, d_n are coefficients defined in [4] and G_n, H_n are Gegenbauer functions of the first and the second kind respectively [13].

In the following figure we present streamlines assuming stream function values equal to $0.1, 0.05, 0.01$, depicted from infinity towards the surface of the inverted prolate spheroid respectively, using only the first term of the series.

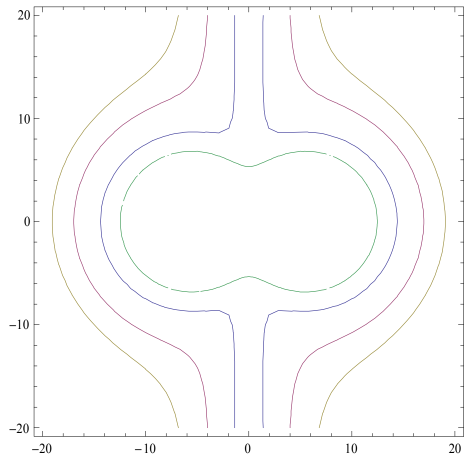

Figure 2. Streamlines in the plane $x_2 = 0$ with $\tau = 1,09$.

III. VELOCITY COMPONENTS, DRAG FORCE, DRAG COEFFICIENT

As it is well known, see for example [9], the velocity components in any coordinate system of revolution (q_1, q_2, ϕ), with radial cylindrical coordinate $\varpi = \varpi(q_1, q_2)$, are given by

$$\upsilon_{q_1} = -\frac{h_2}{\varpi} \frac{\partial \psi}{\partial q_2}, \tag{8}$$

$$\upsilon_{q_2} = \frac{h_1}{\varpi} \frac{\partial \psi}{\partial q_1}, \tag{9}$$

where $\psi = \psi(q_1, q_2, \phi)$ is the stream function and h_1, h_2 are the metric coefficients of the coordinate system.

In the inverted prolate coordinate system the above relations read as

$$\varpi' = \frac{b^2 \sqrt{1-\zeta^2} \sqrt{\tau^2 - 1}}{c(\tau^2 + \zeta^2 - 1)}, \tag{10}$$

$$h_1' = \frac{c(\tau^2 + \zeta^2 - 1)\sqrt{\tau^2 - 1}}{b^2 \sqrt{\tau^2 - \zeta^2}}, \tag{11}$$

$$h_2' = \frac{c(\tau^2 + \zeta^2 - 1)\sqrt{1 - \zeta^2}}{b^2 \sqrt{\tau^2 - \zeta^2}}. \tag{12}$$

Using the stream function (5), the tangential and the normal velocity components become

$$\upsilon_{\tau'} = -\frac{h_2'}{\varpi'} \frac{\partial \psi_\alpha}{\partial \zeta'} = -\frac{h_2'}{\varpi'} \left(\frac{\partial \psi_\alpha}{\partial \zeta} \frac{\partial \zeta}{\partial \zeta'} + \frac{\partial \psi_\alpha}{\partial \tau} \frac{\partial \tau}{\partial \zeta'} \right), \tag{13}$$

$$\upsilon_{\zeta'} = \frac{h_1'}{\varpi'} \frac{\partial \psi_\alpha}{\partial \tau'} = \frac{h_1'}{\varpi'} \left(\frac{\partial \psi_\alpha}{\partial \zeta} \frac{\partial \zeta}{\partial \tau'} + \frac{\partial \psi_\alpha}{\partial \tau} \frac{\partial \tau}{\partial \tau'} \right). \tag{14}$$

Once the velocity field is known, the pressure tensor $\mathbf{\Pi}$ and the shear deformation tensor $\mathbf{\Delta}$ can be also evaluated [9].

Another quantity of interest is the drag force which expresses the resistance to the motion through a medium. Therefore the drag force exerted on the surface of an axially symmetric body [9] is defined as

$$F_z = 8\pi\mu \lim_{r\to\infty} \frac{r(\psi - \psi_\infty)}{\varpi^2}, \tag{15}$$

where μ is the shear viscosity, ψ is the stream function, ψ_∞ is the asymptotic form of the stream function and ϖ is the radial cylindrical coordinate.

In the inverted prolate coordinate system it takes the form

$$F_z = 8\pi\mu \lim_{r'\to\infty} \frac{r'(\psi_\alpha - \psi_\infty)}{\varpi'^2}. \tag{16}$$

After some calculations we obtain

$$\frac{r'(\psi_\alpha - \psi_\infty)}{\varpi'^2} = \frac{b}{c^2}\left[A_2 \frac{G_2(\tau)}{\tau^2-1} + E_2 \frac{G_4(\tau)}{\tau^2-1}\right]\frac{G_2(\zeta)}{1-\zeta^2}$$

$$+\frac{b}{c^2}\sum_{n=2}^{\infty}\left[A_{2n}\frac{G_{2n}(\tau)}{\tau^2-1} + E_{2n}\frac{G_{2n+2}(\tau)}{\tau^2-1}\right]\frac{G_{2n}(\zeta)}{1-\zeta^2}, \tag{17}$$

Therefore

$$F_z = -\frac{4\pi\mu b}{c^2}\left[\frac{3}{4}bcU\ln\frac{\tau_0+1}{\tau_0-1}\right.$$

$$\left.+\sum_{n=2}^{\infty}(A_{2n}+E_{2n})(-1)^{n+1}\frac{1\cdot 3\cdot\ldots\cdot(2n-3)}{2\cdot 4\cdot\ldots\cdot(2n)}\right]. \tag{18}$$

The drag coefficient [9] is then defined by

$$C_D = \frac{2F_z}{\rho U^2 A}, \tag{19}$$

where F_z is the drag force, ρ is the density of the blood plasma, U is the fluid velocity parallel to the x_3-axis in the positive direction and A is the characteristic area. Therefore the drag coefficient is

$$C_D = \frac{-8\mu(\tau_0^2-1)}{\rho U^2 b^3}\left[\frac{3}{4}bcU\ln\frac{\tau_0+1}{\tau_0-1}\right.$$

$$\left.+\sum_{n=2}^{\infty}(A_{2n}+E_{2n})(-1)^{n+1}\frac{1\cdot 3\cdot\ldots\cdot(2n-3)}{2\cdot 4\cdot\ldots\cdot(2n)}\right]. \tag{20}$$

The drag coefficient using only the first term of the series is

$$C_D^{(2)} = \frac{-12\sqrt{\tau_0^2-1}\,\ln\frac{\tau_0+1}{\tau_0-1}}{\text{Re}}, \tag{21}$$

where Re is the Reynolds number, or

$$\text{Re}\, C_D^{(2)} = -12\sqrt{\tau_0^2-1}\,\ln\frac{\tau_0+1}{\tau_0-1}. \tag{22}$$

The negative sign in (22) explains the resistance on the fluid. Next we demonstrate the variation of the values of the absolute value of drag coefficient $\text{Re}\,C_D^{(2)}$ versus τ_0, where $\tau_0 \geq 1$.

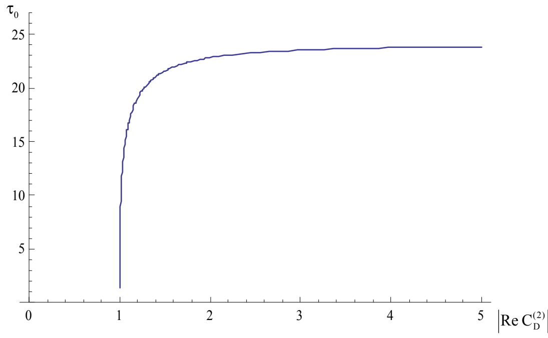

Figure 3. Plot of $\left|\text{Re}\,C_D^{(2)}\right|$ versus τ_0.

It is obvious that as τ_0 increases $\left|\text{Re}\,C_D^{(2)}\right|$ reaches an asymptotic value. This is verified by taking the limit in expression (22). Indeed performing simple calculations we can find that

$$\lim_{\tau_0\to+\infty}\left|\text{Re}\,C_D^{(2)}\right| = 24\,. \tag{23}$$

Summarizing, in the present work we calculated the velocity components, the drag force and the drag coefficient regarding the blood plasma flow past a fixed red blood cell. We used the known stream function describing the creeping flow past an inverted prolate spheroid, in order to model the blood plasma flow past a RBC. Since this is given as a series expansion which convergence fast, we use only the first term of the series to acquire simpler formulas for the drag force and the drag coefficient. This theoretical model, describes the relative flow of the blood plasma with respect to a stationary RBC. It also provides the basis for obtaining analogous results concerning the relative case, where the RBC now moves slowly within a stationary fluid. This relative case is directly applicable to the sedimentation of erythrocytes and specifically in a clinical haematological test, namely the Erythrocyte Sedimentation Rate (ESR). Performing the test, one can measure the rate at which RBCs fall in vitro, under the influence of gravity. This is due to the greater density they have, compared with the density of the blood plasma.

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