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Multidimensional Models for Methodological Validation in Multifractal Analysis

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Abstract: **Multifractal analysis is known as a useful tool in signal analysis. However methods are often used without methodological validation. In this study, we define multidimensional models in order to validate multifractal analysis methods.**

I. INTRODUCTION

Fractal and multifractal analysis provides powerful tools for physical signals analysis. To deal with the only biomedical field, it covers a vast field from the 1D (electrophysiological signals), to the 2D (medical imaging) and the 3D (volume reconstruction from tomographic data).

Thus various methods are often applied with the purpose of classification (normal/abnormal, healthy/pathological,…) : their quantitative results are then exploited in a relative way with a concern of optimizing their discriminative capacities. In addition, their presentation is not always accompanied by a methodological validation.

The purpose of this work is to present models (1D, 2D and 3D), built according to data of the literature and chosen to reflect the characteristics of the real data. These models are then used as "benchmarks testing": the obtained results confront the respective theoretical spectra of these models with those obtained by several methods of multifractal analysis.

II. MODELS GENERATING

In this part, we detail the used multifractal models.

A. 1D Model

We analyzed an analytically solvable example (binomial multiplicative process) [1]. Populations or distributions generated by a multiplicative process have many applications and with the advantage that many properties of their distributions may be easily analyzed (self-similar model).

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Let consider the following multiplicative process, which is generated by dividing the unit interval into two pieces, each of half the previous length, but with unequal measure (say P_1 and $P_2=1-P_1$) and infinitely repeating this process.

Then the measure at the *n*th level of this multiplicative process would consist of $N = 2^n$ of equal length, $L=2^{-n}$ with probabilities $P_i(L) = P_1^{n-k} P_2^k$, k = 0, …, n. The above process is defined as:

$$\tau(q) = \frac{-\ln\left(p_1^q + p_2^q\right)}{\ln(2)} \quad (1)$$

$$\alpha(q) = \tau'(q) = -\frac{p_1^q \ln(p_1) + p_2^q \ln(p_2)}{\left(p_1^q + p_2^q\right)\ln(2)} \quad (2)$$

$$f\left(\alpha(q)\right) = q\alpha(q) - \tau(q) \quad (3)$$

$$\alpha_{\min} = \alpha(+\infty) = \lim_{q\to+\infty} \alpha(q) = \min_i\left(-\frac{\log(P_i)}{\log(2)}\right) \quad (4)$$

$$\alpha_{\max} = \alpha(-\infty) = \lim_{q\to-\infty} \alpha(q) = \max_i\left(-\frac{\log(P_i)}{\log(2)}\right) \quad (5)$$

B. 2D Model

We simulate multifractal singular measurements by using multiplicative cascades models [2, 3]. A simple model is the p-model, which was introduced in fully developed turbulence [4]. It concerns a multiplicative process of conservative cascades, to model statistical scale-invariance properties of dissipation field in a turbulent flow [5]. Its multifractal properties are known analytically.

The principle is as follows: we start with one square defined on the interval $[0\ 1]^2$, then we divide the square into four under squares and we assign randomly to each of them a probability p_i, such as:

$$\sum_i p_i = 1 \quad (6)$$

We iterate this operation on each under square (Fig. 1).

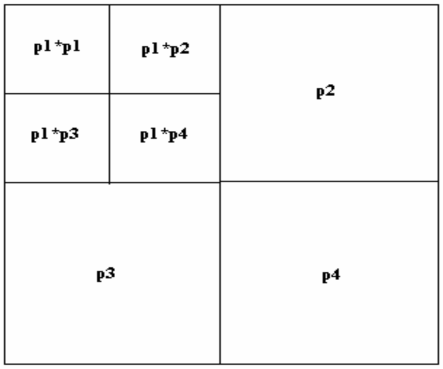


Fig. 1 : 2D p-model

The obtained model is thus self-similar, since the square is divided into four equal parts. The τ(q) function is easily calculable :

$$\tau(q) = -\frac{\log(p_1^q + p_2^q + p_3^q + p_4^q)}{\log(4)} \quad (7)$$

We deduce the α(q) definition :

$$\alpha(q) = \frac{\partial \tau(q)}{\partial q} = -\frac{\sum_{i=1}^{4} p_i^q \log(p_i)}{\log(4)\sum_{i=1}^{4} p_i^q} \quad (8)$$

And by Legendre transform :

$$f(\alpha(q)) = q\alpha(q) - \tau(q) \quad (9)$$

The α limits were calculated in the same manner as (4) and (5) by replacing log(2) by log(4).

C. 3D Model

In 3D, we generated two models, which are the most employed in medical image analysis.

In the first hand we extend the 2D model to 3D model, i.e. we construct a self-similar model scaled by a multinomial measure. We start with one cube defined on the interval $[0\ 1]^3$, then we divide the cube into four under cubes and we randomly assign to each one a probability p_i. We iterate the process on each under cubes. Calculations of the theoretical multifractal spectrum are indentical to 2D case, by taking p_i, for i = 1,…, 8 and by replacing log(4) by log(8).

On the other hand we generate a second model, which is in this time, is self-affine. This model is known in the 2D case and we extend it to 3D case.

We start with one cube defined on the interval $[0\ 1]^3$, we divide it according to three directions, we assign to some boxes a probability p_i and in each one of these boxes we iterate the process. Figure 2 schematizes the construction in the 2D case to help visualization.

In our example, we divide the cube into two in X and Y directions and four in the Z direction, then in the boxes of coordinates I, we affect a corresponding probability P:

I={(0,0,0),(1,0,0),(1,0,1),(0,1,1),(1,0,2),(1,0,3),(1,1,3)}

P=(1/20, 1/15, 1/5, 1/10, 1/5, 1/10, 17/60)

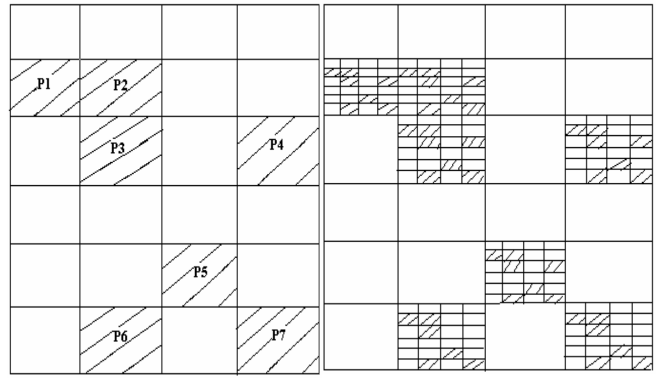


Fig. 2 : 2D Sierpinski model (steps 1 and 2)

It would be too long to detail the calculation of theoretical multifractal spectrum of this model: we will limit our presentation to express the theoretical τ(q) function, which define the multifractal spectrum of a probability [6]. Since the measure is a Bernoulli probability, it comes for the τ formula :

$$\tau(q) = \frac{1}{\log(2)}\log\left(\left(\frac{3}{20}\right)^q 2^{\beta(q)} + \left(\frac{17}{20}\right)^q 2^{\gamma(q)}\right) \quad (10)$$

With :

$$\beta(q) = \frac{1}{\log(2)}\log\left(\left(\frac{1}{3}\right)^q + \left(\frac{2}{3}\right)^q\right) \quad (11)$$

And

$$\gamma(q) = \frac{1}{\log(2)}\log\left(\left(\frac{1}{3}\right)^q 2^{\omega(q)} + \left(\frac{2}{3}\right)^q\right) \quad (12)$$

Where

$$\omega(q) = \frac{1}{\log(4)}\log\left(\left(\frac{3}{17}\right)^q + \left(\frac{2}{17}\right)^q + 2\left(\frac{6}{17}\right)^q\right) \quad (13)$$

III. MULTIFRACTAL SPECTRUM COMPUTING

We use the method defined by Chhabra and Jensen [7] for multifractal spectrum calculation. We will see below that it requires the definition of the Borel measure. We will bring to this method the implementation of a new measure. The method is summarized on the flowchart of figure 3, in the nD case.

The μ measure is defined by the Choquet capacity. In the literature we found many capacities [8,9] with a general definition having the following shape:

$$\mu_i(\delta) = O(x,y)_{\in B_i}\, g(x,y) \quad (14)$$

Where O is an operator dealing with the intensity of pixel g(x,y) in the box i. As an example, the “sum” operator, which is used in Chhabra’s method, or the “max” operator. The main drawback of these operators is their lack of sensitivity to the amplitude or to the spatial distribution of the singularities [10].

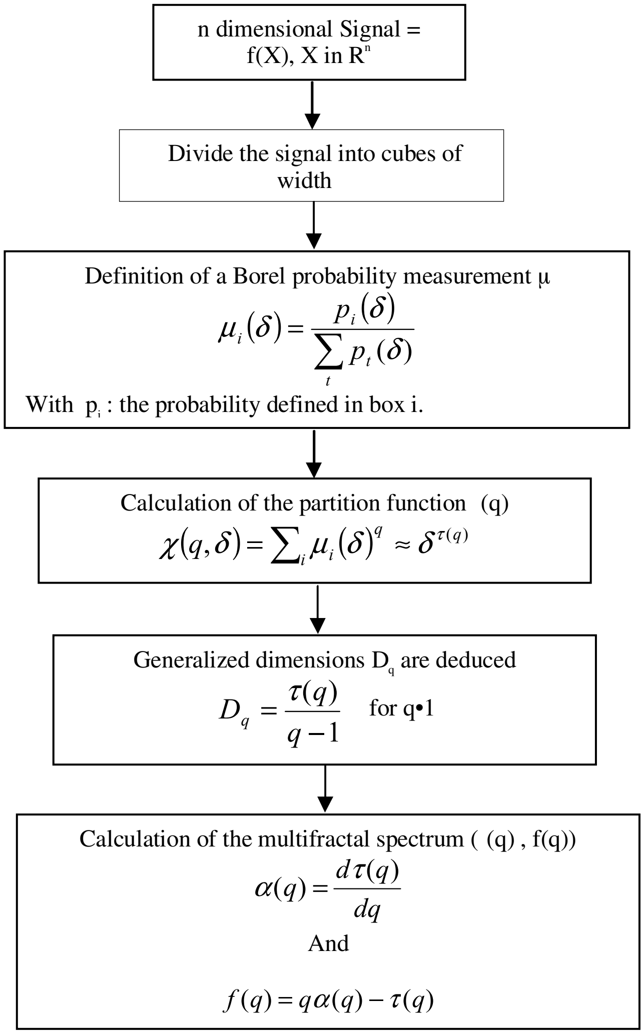


Fig. 3 : Flowchart of the method

We define two new measures for this method. First, we combine one of the previous operators with the gradient "∇" computed on each pixel, defined over n axes and the norm [10]. Thus we obtain three measures which are simultaneously sensitive to amplitude and spatial distribution of the singularities. These measures have the following expression:

$$\mu_{x,i}(\delta) = O\nabla_x g(x,y) \tag{15}$$

$$\mu_{y,i}(\delta) = O\nabla_y g(x,y) \tag{16}$$

$$\mu_i(\delta) = \sqrt{\mu_{x,i}(x,y)^2 + \mu_{y,i}(x,y)^2} \tag{17}$$

With x and y in the box i.

Secondly, a step of the "DBC" method is used to define a measure which we call "DBC-mes", it is defined as follows [11]:

$$p_i(\delta) = \frac{nr_i(\delta)}{Nr} \tag{18}$$

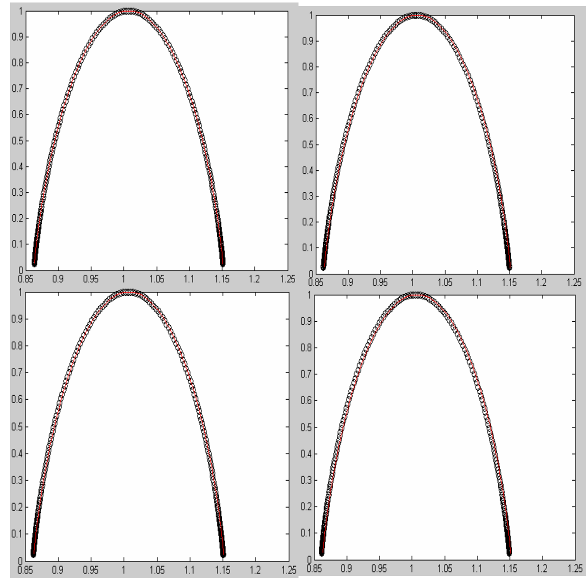
With nr_i : the maximum deviation of gray levels in box i.
Nr : the sum of the deviations of the boxes.

IV. RESULTS

A. 1D case

In the figure 4, we compare Chhabra's method with the analytical solution. We can see that the results are in agreement with the theoretical solution.

We note that in the 1D case, each measure approximates well the theoretical measure (fig. 4). (a)
(b)



(c) (d)

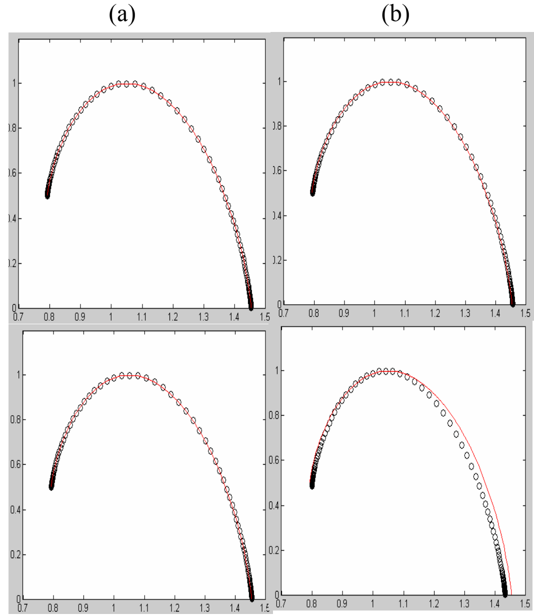
Fig. 4: Representation of the approximated spectrum and the theoretical spectrum (continuous traced on the figures). (a) « sum » measure, (b) « max » measure, (c) « DBC-mes » measure, (d) « gradient » measure.

B. 2D case

We build the model defined in part II-B, with probabilities p_1=1/5, p_2=1/3, p_3=1/3, p_4=2/15.

We note that the « gradient » measure gives the worst approximation of the spectrum, however the result remains correct. For the other measures, the results look very satisfactory (fig. 5).

(a) (b)



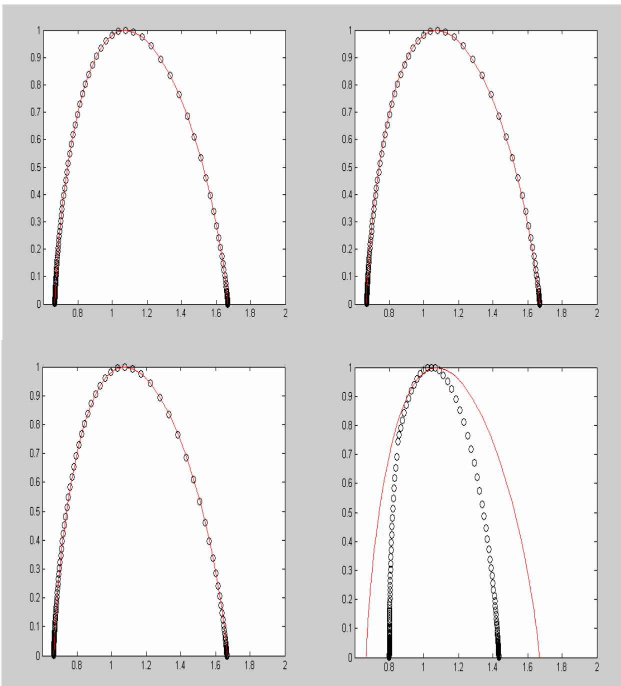
(c) (d)

Fig. 5 : Representation of the approximated spectrum and the theoretical spectrum (continuous traced on the figures). (a) « sum » measure, (b) « max » measure, (c) « DBC-mes » measure, (d) « gradient » measure.

C. 3D case

Concerning the self-similar model, the « gradient » measure is not optimal, whereas the three other measurements approximate the theoretical multifractal spectrum quasi exactly (fig. 6).

(a) (b)



(c) (d)

Fig. 6 : Representation of the approximated spectrum and the theoretical spectrum (continuous traced on the figures). (a) « sum » measure, (b) « max » measure, (c) « DBC-mes » measure, (d) « gradient » measure.

For the self-affine model, the results are completely. different. When we apply the “max” and “gradient” measures, results diverge, and we obtain a concave spectrum which is not a good approximation of the model by using « sum » measure. Finally for « DBC-mes » measure, the results are better than the three other ones, the spectrum width (Δα) is equal to that of the theoretical spectrum (see fig. 7).

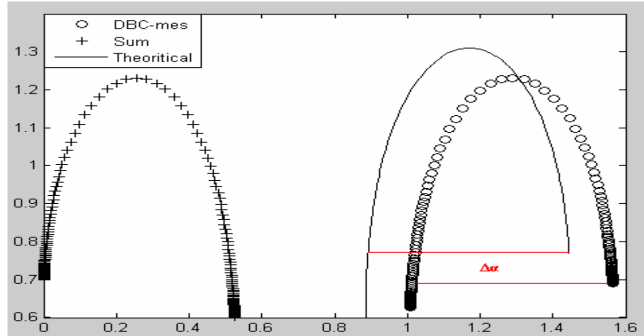


Fig. 7 : A comparison between calculated multifractal spectra by using the « sum » measurements (left) « DBC-mes » measurements (right), and the theoretical spectrum.

V. CONCLUSION

The aim of this work is to find an absolute value of multifractal spectrum, that could be used it in other applications than classification (normal/abnormal). For example in the biomedical domain, fractal and multifractal analysis are often used at ends of classification [12, 13], thereafter we plan to use these analysis for other applications. The second interest is to purpose a theoretical benchmark platform of results confrontation.

Chhabra’s method is powerful for self-similar models, but that is less true for non self-similar models. Using “DBC-mes” measure instead of the « sum » (usually used) measure gives the best results.

Finally it would be interesting to test other methods of calculation of multifractal spectrum, in particular the algorithm based on the wavelet transform (WTMM) [14].

REFERENCES

[1] Niu M, Liang Q, Yu G, Wang F, Yu Z. Multifractal analysis of pressure fluctuation signals in an impinging entrained-flow gasifier. Chemical Engineering and Processing. 2007.

[2] Paladin G, Vulpiani A, Phys. Rep. 156, 148, 1987.

[3] Mandelbrot BB. Fractals and Multifractals : Noise, Turbulence and Galaxies. Vol. 1 of Selecta, 1989.

[4] Meneveau C, Sreenivasan, Phys. Rev. Lett. 59, 1424, 1987.

[5] Meneveau C, Sreenivasan KR, J. Fluid Mech. 224, 429, 1991.

[6] Heurteaux Y. Estimation de la dimension inférieure et de la dimension supérieure des measures. Ann. Inst. H. Poincaré Probab. Statist. 34:309-338, 1998.

[7] Chhabra A, Jensen RV. Direct determination of the *f*(alpha) singularity spectrum. Phys. Rev. Lett. 1989; 62: 1327–30.

[8] Lévy-Véhel J, Mignot P. Multifractal segmentation of images. Fractals, 371-377, 1994.

[9] Berroir JP, Lévy-Véhel J. Multifractal tools for image processing. In Proc. Scandinavian Conference on Image Analysis. Vol. 1, 209-216. 1993.

[10] Abadi M, Grandchamp E. Large deviation spectrum in two dimensions. The international Conference on Signal-Image Technology & Interned-based Systems, SITIS. 2006.

[11] Chaudhuri B, Sarkar N. Texture segmentation using fractal dimension. IEEE Trans. on Pattern Anal. and Machine Intell., 17, 1995.

[12] Lopes R, Dubois P, Dewalle AS, Steinling M, Maouche S, Betrouni N. 3D multifractal analysis of cerebral tomoscintigraphy images, Lecture notes in Computer Assisted Radiology and Surgery, 2007.

[13] Stosic T, Stosic B. multifractal analysis of human retinal vessels. IEEE Transactions on Medical imaging, 2006.

[14] Kestener P., Lina JM, Saint-Jean P, Arneodo A, Wavelet-based multifractal formalism to assist in diagnosis in digitized mammograms. Image Anal. Stereol., 20, 2004.