THE LOW ORDER POLYNOMIAL WINDOW OPTIMIZED WITH VARIOUS CRITERIA AND ITS ABEL TRANSFORM

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Abstract: In 1990 R. M. Lewitt introduced a family of windows and their corresponding Abel transforms based on standard, well-known Kaiser-Bessel window. The Lewitt window has the same drawback as the Kaiser window: high cost of computation because the Bessel function is used. In this paper a low order polynomial window, which has a very simple formula and very low cost of computation is presented. By changing the coefficients of this polynomial window we can obtain time windows with different properties in frequency domain. The polynomial basis functions are compared to blobs based on cosine-form window. The polynomial window and its projection could be utilized in iterative medical image reconstruction.

Introduction

Since 1990 the Lewitt window and its projection are widely used to reconstruction of medical images. We should ask: why? First of all, frequency properties of the Lewitt window are very good. Second, the Abel transform was solved symbolically and third, there is no alternative. Window functions based on cosine function (Hann, Hamming, Blackman etc.) can be computed numerically only. The drawback of the Lewitt function is that it realizes only energetic criterion. When using the Lewitt window there is no possibility to check if other criteria of optimization will be better in medical image reconstruction. The polynomial windows are very suitable to answer this question. It was shown in [3] that polynomial window of high order has almost the same frequency properties as Lewitt window but has much lower computational cost and using different coefficients we realize different optimization criteria. Now, we want to compute low order polynomial window and its Abel transform and compare it with some properties of standard cosine-form windows (Hann, Hamming).

Criteria of optimization

(1) Energetic criterion
The energetic criterion was formulated in the 1960's. It specifies the ratio of the energy contained in sidelobes to the energy of the main lobe. (it is precisely formulated in the work of A. Papoulis [1] as the ratio of signal energy taken in the range $-T..T$ to the energy of the signal in the range $-\infty..\infty$). A window that realizes this criterion is the spheroidal window [2] proposed by D. Slepian et al. The complicated structure of this window led to the search for various simplifications and modifications [3–6], which however were not introducing considerable simplifications. Until today, the best approximation to the spheroidal window is the Kaiser window described for the first time in [7, 8]. This window is designed based on the Bessel function of the first kind and 0th order which may be presented in the form of a geometric series [9]. The introduction of the Bessel function allowed to obtain a window whose width of the main lobe can easily be adjusted and which still retains its very good properties which well optimize the energetic criterion.

(2) Amplitude criterion
The amplitude criterion is a modification of the Dolph-Czebyshev criterion (minimize the level of the sidelobes with simultaneous attainment of the most narrow main lobe). It is based on minimizing sidelobes level for a given width of the main lobe. Its good realization (but not optimal for this width of the main lobe) is the Hamming window [10–12]. It is commonly accepted that the Hamming, Hann, and Blackman windows are the approximations to the DCh window, what is in agreement with the results published by J. C. Burgess in [13].

In 1978 Webster [14, 15] introduced concept of the generalized Hamming window. The generalization is that the cosine is raised to the non-integer power. Time window based on such cosine function (raised to the first power) together with additional coefficients has exactly the same properties as the simple Hamming window.

(3) Sidelobes roll-off criterion
The sidelobes roll-off criterion means that the time window is designed so that the sidelobes would decay as quickly as possible. The attempt to realize this criterion shows that it is impossible to obtain quickly decaying sidelobes with simultaneous low level of the highest sidelobe. This limitation results in the fact that when we wish to obtain the decay greater than $d = -18$ dB/oct., windows with the width of the main lobe greater than WML=2 have to be used, eg. the Blackman window [10, 16–18].

An example of the realization of this criterion for WML=2 (the width of the main lobe is double that

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of the rectangular window) is the Hann window [10, 12, 16, 19–24]. For this window the sidelobe decay is of the order of $d = -18$ dB/oct. with the level of the highest sidelobe equal ca. -31 dB.

Materials and methods

The polynomial window

The main advantage of the polynomial windows family [25] is their very low computational cost and flexibility: many different criteria of optimization can be applied. The formula for the 3rd order polynomial window is shown below (1)

$$w(r) = 1 + a_2\left(\frac{r}{T}\right)^2 + a_3\left(\frac{r}{T}\right)^3 \qquad (1)$$

where

$$r = \sqrt{s^2+t^2}$$

If we would like to calculate a projection of the window we use integration with the formula below

$$AbelT = \int_0^{\sqrt{T^2-s^2}} w(t)\, dt \qquad (2)$$

for (2) we achieve

$$\begin{aligned} Aw(s) &= RT^2\,(12a_3+16a_2+48)+ \qquad (3)\\ &+ RT\,s^2\,(18a_3+32a_2)+ \\ &- \left(\frac{1}{24T^3}\right) 9\,a_3\,s^4\left(\ln(s^2)-2\ln(R+T)\right) \end{aligned}$$

where

$$R = \sqrt{T^2-s^2}$$

We can observe that the formula (4) has a very simple form and low cost of computation. By changing the coefficients of the polynomial window we can obtain time windows with different properties in frequency domain (EC – energetic criterion, AC – amplitude criterion, SC – sidelobes decay criterion).

Coeff	EC	AC	SC
a2	-2.876	-2.796	-3
a3	1.917	1.864	2

Table 1: Coefficients for various criteria

The cosine form window

The Hann and Hamming window are widely used in general DSP applications. Such a window can be described with the following formula:

$$w(t) = a + b\,cos\left(\frac{2\pi t}{T}\right) \qquad (4)$$

where a and b are coefficients varying with the optimization criterion; $t = -T..T$; 2T – width of the window. If we need to set $w(0)$ equal to 1 a, b must realize: $b = 1-a$.

Unfortunately, the Abel transform of a cosine-form window can not be computed algebraically:

$$AbelHann = \int_0^{\sqrt{T^2-s^2}} a + b\,\cos\left(\frac{2\pi t}{T}\right)\, dt \qquad (5)$$

because $\int_0^{\sqrt{T^2-s^2}} \cos\left(\frac{2\pi t}{T}\right)\, dt$ can be find numerically only. This is a big disadvantage of cosine form windows (Hann, Hanning, Blackman, Nuttall etc).
The following short MAPLE V program

```
> restart:
> T:=100;
> r:=sqrt(s^2+t^2):
> Okno:=r->a+(1-a)*cos(Pi*r/T);
> ps:=2*int(Okno(r),t=0..sqrt(T^2-s^2));
> a:=0.5; %Hann window
> Tot:=eval(Okno(0)+2*Okno(T/2))*T;%norm
> for s from 0 by 1 to T do
    evalf(ps)/Tot
  end do;
```

yields a vector that contains values of the Hann Abel transform. Table 2 shows that coefficients of cosine-form window depend on criteria of optimization. In the amplitude criterion case rounded values are preferred $a = 0.54$, $b = 0.46$.

Criteria	a	b
AC	25/46	42/92
SC	0.5	0.5
EC	0.5229	0.4771

Table 2: Coefficients (4) for different criteria of optimization

Results

In the Figure 1, we can observe one half of the polynomial window and modified cosine window (energetic criterion) in the time domain (Figure 1(a)) and in the frequency domain (Figure 1(b)). These windows are slightly different; most sidelobes of 3rd degree polynomial window are a slightly lower than for the cosine window, but the cost of computation of the polynomial window is 2.67 times lower [25]. Generally speaking, both the time and the frequency properties are similar. If we compare ART reconstruction of the well known Shepp-Logan phantom using the polynomial and cosine windows (Figure 1(c) and 1(d)) we may realize that are very similar and both reconstructions preserve all the necessary details.

The amplitude criterion (Figure 2) tries to achieve the level of the sidelobes as low as possible for a fixed width of the main lobe. We observe that the highest sidelobe

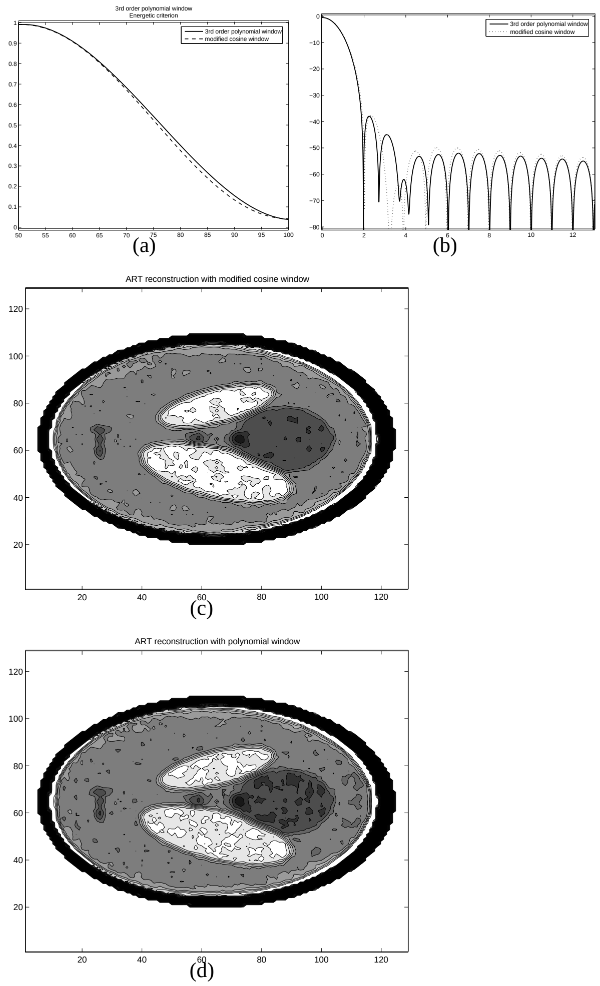

Figure 1: Criterion of optimization EC

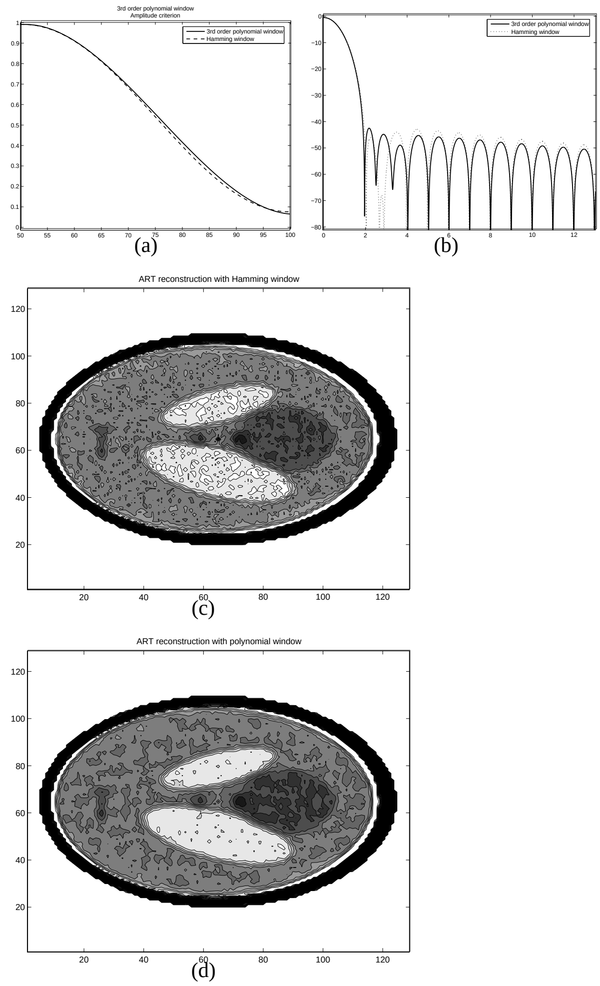

Figure 2: Criterion of optimization AC

has -43.6 dB for the 3rd order polynomial window and -42.6 dB for the Hamming window (Figure 2(b)). In this case the polynomial window is faster and slightly better. The ART reconstructions (Figure 2(c) and 2(d)) are also similar.

The most famous of cosine-form windows that realize sidelobes decay criterion (Figure 3) is the Hann window (the width of the main lobe equal to double the width of the rectangular window). Both the polynomial window and the Hann window have similar decay of sidelobes (Figure 3(b)). We can observe that the sidelobes of the polynomial window have less regular decay. This is a small drawback. Properties of reconstruction images (Figure 3(b) and 3(c)) are very similar; in the polynomial window case the reconstructed image consist a little bit more visible details.

Conclusions

The low order polynomial window family which have similar properties in the time and frequency domains like the Hann and Hamming windows and their corresponding Abel transforms were investigated in this work. The simple formula of the polynomial window gives also a simple formula of its Abel transform for 2D degree polynomial window. By changing the coefficients of the polynomial we achieve very flexible window and its projection (Abel transform), which could be utilized in iterative medical image reconstruction.

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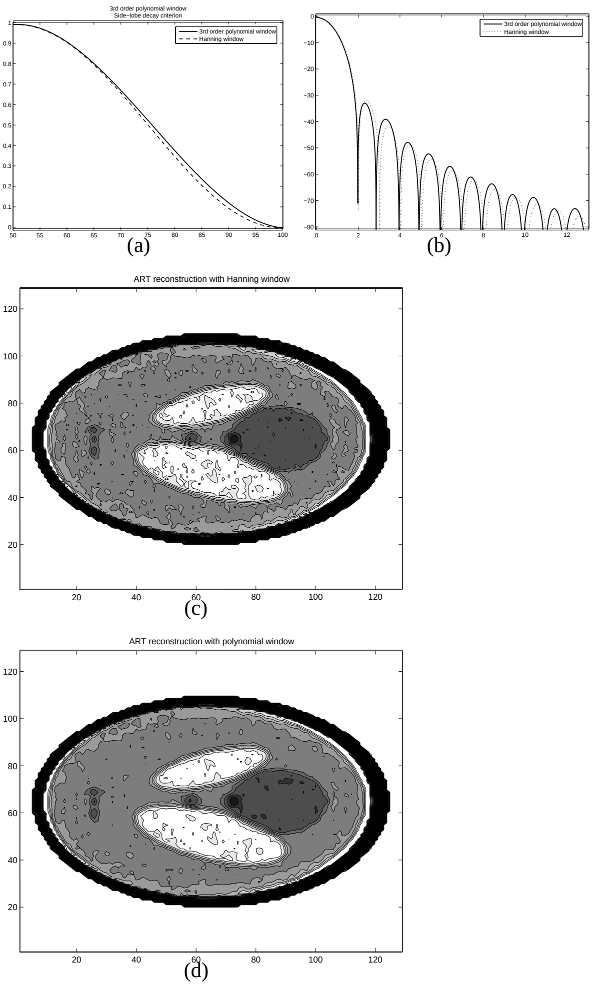

Figure 3: Criterion of optimization SC

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