ABEL TRANSFORM OF POLYNOMIAL WINDOW WITH ODD AND EVEN COEFFICIENT

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Abstract: The polynomial window which has a very simple formula and very low cost of computation can be widely used in DSP algorithms, particularly during medical image reconstructions. The results are very promising: the polynomial window is faster than competitors and its parameters are similar in time and frequency domain. The Abel transform in general form of polynomial window is also presented. The polynomial window and its projection could be utilized in iterative image reconstruction.

Introduction

In 1990 R. M. Lewitt introduced a family of windows and their corresponding Abel transforms based on standard, well-known Kaiser-Bessel window. The Lewitt window has the same drawback that the Kaiser window has: high cost of computation because the Bessel function is used during its calculation. In this paper the polynomial window ([1]) which has a very simple formula and very low cost of computation is presented. By changing the coefficients of polynomial window we can obtain time windows with different properties in the frequency domain. The Abel transform in general form of polynomial window is also presented. The polynomial window and its projection could be utilized in iterative image reconstruction.

Materials and Methods

The problem which should be solved is to find the symbolic Abel transform of a given polynomial window. The polynomial window in its general form [1] is described by the following equation (1):

$$p(t) = 1 + \sum_{k=1}^{N} a_k \left(\frac{t}{T}\right)^k \tag{1}$$

The formula of the polynomial window can be broken down into two sums that contain the odd and the even coefficients respectively.

$$p(t) = 1 + \sum_{k=1}^{ceil(1/2N)} a_{2k-1}\left(\frac{t}{T}\right)^{2k-1} + \sum_{k=1}^{floor(1/2N)} a_{2k}\left(\frac{t}{T}\right)^{2k} \tag{2}$$

let

$$r = \sqrt{s^2+t^2} \tag{3}$$

and

$$AbT(s) = 2\int_0^z p(r)\,dt \tag{4}$$

where $z = \sqrt{T^2 - s^2}$

Analyzing the formula (2) we can observe that this equation can be divided into three elements (5):

$$\begin{aligned} AbP(s) &= 2\int_0^z 1\,dt + \\ &+ 2\int_0^z \left(\sum_{k=1}^{ceil(1/2N)} a_{2k-1}\left(\frac{r}{T}\right)^{2k-1}\right) dt + \\ &+ 2\int_0^z \left(\sum_{k=1}^{floor(1/2N)} a_{2k}\left(\frac{r}{T}\right)^{2k} dt\right) \end{aligned} \tag{5}$$

interchanging the integration and summation operators we obtain:

$$\begin{aligned} AbP(s) &= 2\int_0^z 1dt + \\ &+ \underbrace{2\sum_{k=1}^{ceil(1/2N)} \left(\int_0^z a_{2k-1}\left(\frac{r}{T}\right)^{2k-1} dt\right)}_{\text{odd}} + \\ &+ \underbrace{2\sum_{k=1}^{floor(1/2N)} \left(\int_0^z a_{2k}\left(\frac{r}{T}\right)^{2k} dt\right)}_{\text{even}} \end{aligned} \tag{6}$$

From the form of the equation (6) we can draw a simple conclusion that in order to compute the complete Abel transform, it is sufficient to determine a method of calculating the odd and the even order polynomials.

1 Abel transform of even elements

Let us consider then even-order polynomials first:

$$2\sum_{k=1}^{floor(1/2N)} \left(\int_0^z a_{2k}\left(\frac{\sqrt{s^2+t^2}}{T}\right)^{2k} dt\right) \tag{7}$$

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It is necessary to notice that a_{2k} and T in denominator do not depend on the integration variable t. Therefore, to simplify, we remove them from equation (7). Our main goal to find general form of the Abel transform of even elements is reduced to find a solution to the integral (8):

$$AbT(s,k) = 2\int_0^z \left(\sqrt{s^2+t^2}\right)^{2k} dt \qquad (8)$$

$$AbT(s,1) = \underline{2\,z} \qquad \text{for} \quad k=1 \qquad (9)$$

$$AbT(s,2) = \underline{2\,z}s^4 + \underbrace{\frac{4}{\mathbf{3}}z^3s^2}+\frac{2}{5}z^5 \qquad \text{for} \quad k=2 \qquad (10)$$

$$AbT(s,4) = \underline{2\,z}s^8 + \underbrace{\frac{8}{3}z^3s^6+\frac{12}{\mathbf{5}}z^5s^4+\frac{8}{7}z^7s^2}+\frac{2}{9}z^9 \qquad \text{for} \quad k=3 \qquad (11)$$

$$AbT(s,6) = \underline{2\,z}s^{12} + \underbrace{\frac{12}{3}z^3s^{10}+\frac{30}{5}z^5s^8+\frac{40}{\mathbf{7}}z^7s^6+\frac{30}{9}z^9s^4+\frac{12}{11}z^{11}s^2}+\frac{2}{13}z^{13} \qquad \text{for} \quad k=4 \qquad (12)$$

$$AbT(s,8) = \underline{2\,z}s^{16} + \underbrace{\frac{16}{3}z^3s^{14}+\frac{56}{5}z^5s^{12}+\frac{112}{7}z^7s^{10}+\frac{140}{\mathbf{9}}z^9s^8+\frac{112}{11}z^{11}s^6+\frac{56}{13}z^{13}s^4+\frac{16}{15}z^{15}s^2}+\frac{2}{17}z^{17} \qquad \text{for} \quad k=5 \qquad (13)$$

We can observe a relationship. Equations from (9)..(13) contain three parts. First part $\underline{2\,z}s^{2k}$ has a constant element (underlined) and a variable degree of power of s dependent on k. The second part, which is at the end of the expression shown (9)..(13) has a constant form with degree of power and value of coefficient dependent on k: $z^{2k+1}\frac{2}{2k+1}$. The third, central part, has form of series where degree of power of z and s and coefficient in $z^x \cdot s^y$ are variable. These coefficients have such a form that their denominator is formed by the sequence $3,5,7,...2l-1$ and their numerator contains coefficient fellows from equation $2\prod_{l=0}^{k-l}\frac{k-l}{l+1}$. To summarize, the Abel transform of the even order can be described by formula (14)

$$AbEven(s,k) = \frac{a_{2k}}{T^{2k}}\left(2zs^{2k}+z^{2k+1}\frac{2}{2k+1}+\left(\sum_{l=2}^{k}\frac{2\prod_{l=0}^{k-l}\frac{k-l}{l+1}}{2l-1}z^{2l-1}\cdot s^{2k-2l+2}\right)\right) \qquad (14)$$

2 Abel transform of odd elements

Let us consider then odd part of (6)

$$2\sum_{k=1}^{ceil(1/2N)}\left(\int_0^z a_{2k-1}\left(\frac{\sqrt{s^2+t^2}}{T}\right)^{2k-1}dt\right) \qquad (15)$$

Both a_{2k-1} and T are not dependent on integration variable t. Similarly like in even case we remove them from our equation. Our main goal to find general form of Abel transform of odd elements resolve itself to find solution for integral (16):

$$AbT(s,k) = 2\int_0^z \left(\sqrt{s^2+t^2}\right)^{2k-1} dt \qquad (16)$$

We can solve above integral for $k = 1..5$

k	formula
1	$-1/2\,s^2\ln\left(s^2\right)+z\sqrt{L}+s^2\ln\left(z+\sqrt{L}\right)$
2	$-3/8\,s^4\ln\left(s^2\right)+1/2\,zL^{3/2}+3/4\,s^2z\sqrt{L}+3/4\,s^4\ln\left(z+\sqrt{L}\right)$
3	$-\frac{5}{16}s^6\ln\left(s^2\right)+1/3\,zL^{5/2}+\frac{5}{12}s^2zL^{3/2}+5/8\,s^4z\sqrt{L}+5/8\,s^6\ln\left(z+\sqrt{L}\right)$
4	$-\frac{35}{128}s^8\ln\left(s^2\right)+1/4\,zL^{7/2}+\frac{7}{24}s^2zL^{5/2}+\frac{35}{96}s^4zL^{3/2}+\frac{35}{64}s^6z\sqrt{L}+\frac{35}{64}s^8\ln\left(z+\sqrt{L}\right)$
5	$-\frac{63}{256}s^{10}\ln\left(s^2\right)+1/5\,zL^{9/2}+\frac{9}{40}s^2zL^{7/2}+\frac{21}{80}s^4zL^{5/2}+\frac{21}{64}s^6zL^{3/2}+\frac{63}{128}s^8z\sqrt{L}+\frac{63}{128}s^{10}\ln\left(z+\sqrt{L}\right)$ (17)

where $L = s^2+z^2$

Based on (17) we can conclude that:

(1) it is possible to distinguish three parts:
- $-d\,s^x\ln\left(s^2\right)$ expression contains: minus sign, coefficient d which depend on N, s^x (degree of power depend on N) and constant element: $\ln\left(s^2\right)$
- $+c\,s^x\ln\left(z+\sqrt{L}\right)$ the element at the end of for-

mula has ordered form: sign $+$, coefficient c (dependent on N), s^x where x depend on N and constant element $\ln\left(z+\sqrt{L}\right)$

We can notice that:

- $d = 1/2c$
- s^x occure in both parts, and $x = 2k$
- we have to find method how to solve coefficient c

- there is third part in equation (17), where number of elements are depend on k and has form of series: $z\sum \mathrm{c}\cdot s^? L^?$ where c is a coefficient varying depend on k

(2) based on above conclusions it is necessary to find rule describe coefficient c and formula $z\sum \mathrm{n}\cdot s^? L^?$

First of all we consider above formula. Coefficient n occur with highest degree of power of s^{2k} for each k. We can notice that formula $1/k$ can be found for $s = 0$; it is worth checking if following coefficients they are multiplicity of $1/k$. To check this property we divide next coefficient by $1/k$. Lets analyze situation for $k = 7$.

numer. $\frac{n_{l+1}}{n_l}$	denom. $\frac{n_{l+1}}{n_l}$	numer. ratio	denom. ratio	proper ratio value		proper numer./denom. value	
13/1	84/7	13	12	13	12	13/1	84/7
143/13	840/84	11	10	11	10	143/13	840/84
429/143	2240/840	3	8/3	9	7	1287/143	6720/840
143/429	640/2240	1/3	2/7	7	6	9009/1287	40320/6720
143/143	512/640	1	4/5	5	4	45045/9009	161280/40320
429/143	1024/512	3	2	3	2	135135/45045	322560/161280

The first two column consist reduce values of numerator and denominator. Last two column are composed with not reduce values of coefficients. Analyzing these values we can assume proper rule which can be describe with:

$$n_{k,m} = \prod_{l=0}^{k-m-2} \frac{(2k-1)/2-l}{k-1-l} \tag{18}$$

where variable m cause generating values of following coefficient for particular degree of power of L and s. Thus, we can write:

$$AbOdd_1 = \frac{z}{k}\sum_{m=0}^{k-1}\left(n_{k,m}\cdot L^{\frac{1}{2}+m} s^{2(k-1-m)}\right) \tag{19}$$

The last task is to find rule for calculate coefficient c. We know that coefficient of highest degree of power of s in formula (19) is equal c (this is also the same situation when $m = 0$ in (18)). To check this preposition we calculate values of c for $k = 1..7$

k	coefficient
1	1
2	3/4
3	5/8
4	35/64
5	63/128
6	231/512
7	429/1024

We can notice that numerator and denominator have some regularity:

$$\frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\cdots}{1\cdot 4\cdot 6\cdot 8\cdot 10\cdots} \tag{20}$$

what can be generalize to

$$n_{k,0} = \frac{1}{k}\prod_{l=0}^{k-2}\frac{k-1/2-l}{k-1-l} \tag{21}$$

After adding all elements of formula we achieve:

$$AbOdd_2 = n_k s^{2k}\left(\ln\left(z+\sqrt{L}\right) - 1/2\ln\left(s^2\right)\right) \tag{22}$$

Therefore for formula (16) we can write general rule:

$$\frac{1}{k}\left(z\sum_{m=0}^{k-1}\prod_{l=0}^{k-2-m}\frac{k-1/2-l}{k-1-l}L^{1/2+m}s^{2k-2-2m} + \prod_{l=0}^{k-2}\frac{k-1/2-l}{k-1-l}s^{2k}\left(\ln\left(z+\sqrt{L}\right)-1/2\ln\left(s^2\right)\right)\right) \tag{23}$$

The Abel transform of odd elements can be describe with formula:

$$AbOdd(s,k) = 2\int_0^z a_{2k-1}\left(\frac{\sqrt{s^2+t^2}}{T}\right)^{2k-1} dt \tag{24}$$

$$AbOdd(s,k) = \frac{a_{2k-1}}{kT^{2k-1}}\left(z\sum_{m=0}^{k-1} n_{k,m}L^{1/2+m}s^{2k-2-2m} + n_{k,0}s^{2k}\left(\ln\left(z+\sqrt{L}\right)-1/2\ln\left(s^2\right)\right)\right) \tag{25}$$

where $n_{k,m}$ has form (18)

3 The Maple procedure of Abel transform

Below there is a MAPLE V procedure which solve Abel transform of A-degree polynomial window elements.

```
Abel:=proc(A)
global k,L;
local W,z;
if type(A,odd) then k:=(A+1)/2 else k:=A/2 end if;
z:=sqrt(T^2-s^2);L:=(s^2+z^2);
if type(A,odd) then
 W:=a[2*k-1]/(k*T^(2*k-1))*...
  (z*sum(product((((2*k-1)/2-l)/(k-1-l)),l=0..k-2-m)*...
  L^((1/2+m))*s^(2*(k-1-m)),m=0..k-1)+
  +(product(((2*k-1)/2-l)/(k-1-l), l = 0 .. k-2))*...
  s^(2*k)*simplify(ln(z+L^(1/2))+ln(1/(s^2)^(1/2))))
end;
if type(A,even) then
  if A=0 then W:=2*z else
   W:=a[2*k]/T^(2*k)*(2*z*s^(2*k)+2/(2*k+1)*z^(2*k+1)+
   +sum((2*product((k-l)/(l+1),l=0..k-l))/(2*l-1)*...
   z^(2*l-1)*s^(2*(k-l)+2),l=2..k))
  end;
end;
simplify(W) assuming T::positive;
end proc:
```

Lets find Abel transform of 3rd degree polynomial window: we need zero, second and third order of polynomial. Using `Proj:=Abel(0)+Abel(2)+Abel(3);` gives:

$$
\begin{aligned}
Proj &= 2z+\frac{2}{3}\,\frac{a_2 z\,\left(2s^2+T^2\right)}{T^2}+\\
&+ 1/8\,\frac{a_3\left(6z\,Ts^2+4z\,T^3\right)}{T^3}+\\
&+ 1/8\,\frac{a_3\left(6s^4\ln(z+T)-3s^4\ln\left(s^2\right)\right)}{T^3} \qquad (26)
\end{aligned}
$$

where $z=\sqrt{T^2-s^2}$

To make sure that procedure Abel is correct lets compute transform Abel directly.

$$
w := (t,A) \mapsto 2\frac{a_A}{T^A}\int_0^{\sqrt{T^2-s^2}} \sqrt{s^2+t^2}^A \qquad (27)
$$

```
OProj:=proc(m)
global k;local W,w;
w:=(t,A)->(sqrt((s^2+t^2))^(A))/T^A*a[A]:
if type(m,odd) then k:=(m+1)/2 else k:=m/2 end if;
if type(m,odd)
then  W:=2*int(w(t,2*k-1),t=0..sqrt(T^2-s^2))
end;
   if type(m,even) then
      if m=0 then W:=2*int(1,t=0..sqrt(T^2-s^2))
         else W:=2*int(w(t,2*k),t=0..sqrt(T^2-s^2))
      end;
   end;
simplify(W) assuming T::positive;
end proc:
```

After: `OrgProj:=OProj(0)+OProj(2)+ OProj(3);` we achieve

$$
\begin{aligned}
OrgProj &= 2z+\frac{2}{3}\,\frac{a_2 z\left(2s^2+T^2\right)}{T^2}+\\
&+ \frac{2}{8}\,\frac{a_3\left(3z\,Ts^2+2z\,T^3\right)}{T^3}+\\
&+ \frac{3}{8}\,\frac{a_3\left(2s^4\ln(z+T)-s^4\ln\left(s^2\right)\right)}{T^3} \qquad (28)
\end{aligned}
$$

where $z=\sqrt{T^2-s^2}$

Lets simplify both expression:
`simplify(Proj-OrgProj);`
result is equal zero.

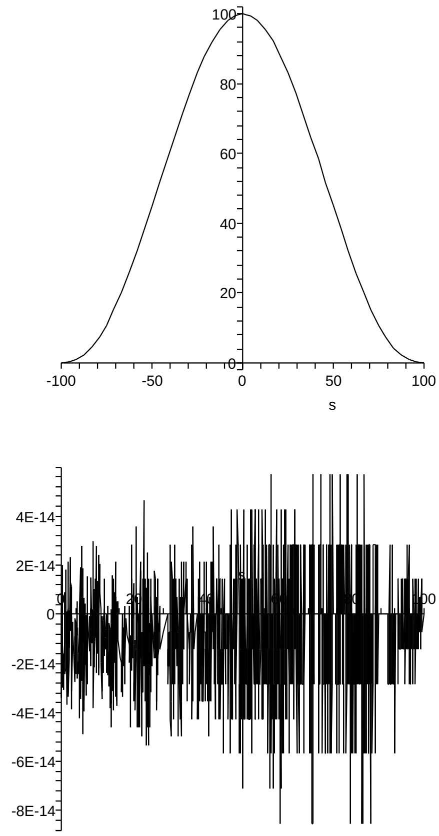

Figure 1: The polynomial window optimized with energetic criterion (upper) and error between `Proj` and `OrgProj`.

Conclusions

The polynomial window which has a very simple formula and very low cost of computation can be widely used in DSP algorithms, particularly during medical image reconstructions. The results are very promising: the polynomial window is faster than competitors and its parameters are similar in time and frequency domain. The Abel transform in general form of polynomial window is also presented. The polynomial window and its projection could be utilized in iterative image reconstruction.

References

[1] M. JASKULA. *Fast time window in dsp*. PhD thesis, Szczecin University of Technology, Szczecin, Poland, June 1999.

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